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SINGULAR INTEGRAL EQUATIONS IN ELASTIC DIELECTRICS

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Abstract—Galerkin representations for the displacement vector, polarization vector and the potential field are obtained by elementary matrix inversions of the equations of equilibrium. Matrices of fundamental solutions of an infinite elastic dielectric continuum subjected to a concentrated body force, an electric force, and a charge density, are constructed. Theorems are proved on the discontinuity of double layer potentials and R, M, Y operators of single layer potentials. By means of these theorems, the solution of the two basic boundary value problems has been reduced to the solution of a system of seven singular integral equations.

1. INTRODUCTION

Recent research work by Mindlin[1, 2] in linear elastic dielectrics includes the polarization gradient in the stored energy density function and is intended to bring together the classical theory of piezoelectricity and Toupin's[3] equations of elastic dielectrics. This extension accommodates several observed phenomena otherwise not included: an electro-mechanical interaction in isotropic centro-symmetric and non-symmetric materials, surface energy of deformation and polarization, capacitance of thin dielectric films, acoustical activity and optical activity when the magnetic field is also included. The polarization gradient supplies terms found in long wave limits of finite difference equations of lattice theories of crystals[1].

The solutions to the equilibrium equations of the Mindlin theory of linear elastic dielectrics with polarization gradient in terms of functions analogous to Papkovitch-type functions of classical elasticity have been constructed by Schwarz[4]. The solution is then employed to solve the problem of a concentrated force applied at a point in an infinite elastic dielectric continuum.

Singular integral equations of coupled thermoelasticity have been obtained by Ignaczak and Nowacki[11], of classical elasticity by Kupradze[5], and of micropolar thermo-elasticity by Shanker[10].

In this paper, by an elementary matrix inversion of equations of equilibrium of linear elastic dielectrics with polarization gradient, we obtain Galerkin representations for the displacement vector, polarization vector and the potential of the Maxwell field. In Section 4, the matrices of fundamental solutions of an unbounded isotropic elastic dielectric continuum, subjected to a concentrated body force, electric force and charge density are constructed. As in Kelvin's solution of classical elasticity, these singularities are found to be of order $1/r$. In Section 5, the surface potentials of single and double layer are introduced and discontinuity theorems of the double layer potentials and R, M, Y operators of single layer potentials are stated and proved for Hölder class of density functions. Furthermore, these theorems have been utilized to formulate the first two basic boundary value problems of linear elastic dielectrics in the form of a system of seven singular integral equations. The symbolic determinant for the first interior boundary value problem is found to be non-zero. It is concluded that the system can be regularized and can thus be solved for the density functions.

2. THE BASIC EQUATIONS

Let a homogeneous isotropic elastic dielectric occupy a region V in a rectangular Cartesian coordinate system whose boundary S separates it from an outer vacuum V'.

The basic equations developed in [2] reduce to the equations of equilibrium,

$$T_{ij,i} + f_j = 0, \qquad \bar{E}_j + E_{ij,i} - \phi_{,j} + E_{0j} = 0 \tag{2.1}$$

$$-\epsilon_0 \phi_{,ii} + P_{i,i} = -\rho_c \text{ in } V, \qquad \phi_{,ii} = 0 \text{ in } V' \tag{2.2}$$

the kinematic relations,

$$E_i^{MS} = -\phi_{,i}, \qquad S_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.3}$$

the constitutive laws,

$$\begin{aligned} -\bar{E}_j &= aP_j \\ E_{ij} &= b_{12}\delta_{ij}P_{k,k} + (b_{44} + b_{77})P_{j,i} + (b_{44} - b_{77})P_{i,j} + d_{12}\delta_{ij}S_{kk} + 2d_{44}S_{ij} + b_0\delta_{ij} \\ T_{ij} &= T_{ji} = d_{12}\delta_{ij}P_{k,k} + d_{44}(P_{j,i} + P_{i,j}) + c_{12}\delta_{ij}S_{kk} + 2c_{44}S_{ij} \end{aligned} \tag{2.4}$$

and the boundary conditions,

$$\begin{aligned} n_i T_{ij} &= k_j, \quad n_i E_{ij} = S_j \\ n_i[P_i - \epsilon_0 \| \phi_{,i} \|] &= \theta(x) \end{aligned} \tag{2.5}$$

in which we use the following notations: T_{ij}—the stress tensor components, E_{ij}—the electric tensor components, S_{ij}—the strain tensor components, u_i—the components of the displacement vector, P_i—the components of a polarization vector, $P_{i,j}$—the components of polarization gradient tensor, E_i—the components of the local electric force vector, E_i^{MS}—the components of the Maxwell self-field vector, f_i—the components of external body force vector, n_i—the components of the unit outward normal vector, ϕ—the potential of the Maxwell field, $\|\phi_{,i}\|$—jump in $\phi_{,i}$ across S, $k_1(x)$, $S_i(x)$, $\theta_i(x)$—the surface loadings, ϵ_0—the permittivity of vacuum, ρ_c—the charge density and a, b_{12}, b_{44}, b_{77}, c_{12}, c_{44}, d_{12}, d_{44} are material constants.

From Eqns. (2.1–2.5), eliminating $\bar{E}_j$, E_{ij} and T_{ij}, the following system of basic equations are obtained

$$\begin{aligned} [c_{44}\nabla^2 + (c_{12} + c_{44})\nabla\nabla.]\bar{u} + [d_{44}\nabla^2 + (d_{12} + d_{44})\nabla\nabla.]\bar{P} &= -\bar{f}, \\ [d_{44}\nabla^2 + (d_{12} + d_{44})\nabla\nabla.]\bar{u} + [(b_{44} + b_{77})\nabla^2 + (b_{12} + b_{44} - b_{77})\nabla\nabla.]\bar{P} - \nabla\phi - a\bar{P} &= -\bar{E}_0 \\ \nabla.\bar{P} - \epsilon_0\nabla^2\phi = -\rho_c \text{ in } V, \quad \nabla^2\phi &= 0 \text{ in } V' \end{aligned} \tag{2.6}$$

together with boundary conditions in the form

$$\begin{aligned} K_{(c)}\bar{u} + K_{(d)}\bar{P} &= R(\bar{u},\bar{P}) = \bar{k}_{(x)} \\ K_{(d)}\bar{u} + K_{(b)}\bar{P} - b_{77}\bar{n} \times \nabla \times \bar{P} + b_0\bar{n} &= M(\bar{u},\bar{P}) = \bar{S}_{(x)} \\ \bar{n}.[\bar{P} - \epsilon_0\|\nabla\phi\|] &= Y(\bar{P},\phi) = \theta_{(x)} \end{aligned} \tag{2.7}$$

where R, M, Y and $K_{(x)}$ are the surface operators and

$$K_{(x)} = x_{12}\bar{n}\nabla. + 2x_{44}\bar{n}.\nabla + x_{44}\bar{n} \times \nabla \times x = b,c,d.$$

3. GALERKIN'S REPRESENTATION

Let

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i}\,(i = 1,2,3), \quad q = X_1^2 + X_2^2 + X_3^2 \\ q_i &= c_{44}q, \qquad\qquad q_2 = (b_{44} + b_{77})q - a \\ x &= x_{12} + 2x_{44}\,(x = b,c,d) \end{aligned} \tag{3.1}$$

and L be the matrix associated with the system (2.6). It is clear that the system (2.6) is equivalent to the matrix equation

$$L\begin{bmatrix} \bar{u} \\ \bar{P} \\ \phi \end{bmatrix} = -\begin{bmatrix} \bar{f} \\ \bar{E}_0 \\ p_c \end{bmatrix} \tag{3.2}$$

where

$$L = \begin{bmatrix} c_{44}qI + (c - c_{44})Z & d_{44}qI + (d - d_{44})Z & 0 \\ d_{44}qI + (d - d_{44})Z & q_2I + (b - b_{44} - b_{77})Z & -X \\ 0^t & X^t & -\epsilon_0 q \end{bmatrix} \tag{3.3}$$

where $\bar{u}$ and $\bar{P}$ stand for the column vectors $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$ respectively, the matrices I, X, Z are defined by

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad Z = \begin{bmatrix} X_1^2 & X_1X_2 & X_1X_3 \\ X_2X_1 & X_2^2 & X_2X_3 \\ X_3X_1 & X_3X_2 & X_3^2 \end{bmatrix} \tag{3.4}$$

and the superscript t over a matrix denotes its transpose. Making use of the results

$$ZX = qX \quad Z^2 = qZ \quad X^tX = q \quad XX^t = Z \quad X^tZ = qX^t \tag{3.5}$$

the inverse of L is found to be

$$L^{-1} = \begin{bmatrix} \dfrac{-q_2q\Delta_1 I + \Delta_3 Z}{q^2\Delta_1\Delta_2} & \dfrac{d_{44}q\Delta_1 I - \Delta_4 Z}{q\Delta_1\Delta_2} & -\dfrac{dX}{q\Delta_1} \\ \dfrac{d_{44}q^2\Delta_1 I - q\Delta_4 Z}{q^2\Delta_1\Delta_2} & \dfrac{-c_{44}q\Delta_1 I + \Delta_5 Z}{q\Delta_1\Delta_2} & \dfrac{cX}{q\Delta_1} \\ \dfrac{dq\Delta_2 X^t}{q^2\Delta_1\Delta_2} & \dfrac{-c\Delta_2 X^t}{q\Delta_1\Delta_2} & \dfrac{(bc - d^2)q - ac}{q\Delta_1} \end{bmatrix} \tag{3.6}$$

where

$$
\begin{aligned}
\Delta_1 &= c(1+\epsilon_0 a)+\epsilon_0(d^2-bc)q \\
\Delta_2 &= d_{44}^2 q - c_{44}q_2 \\
\Delta_3 &= (1+\epsilon_0 a - \epsilon_0 bq)\Delta_2 + q_2\Delta_1 \\
\Delta_4 &= d_{44}\Delta_1 - \epsilon_0 d\Delta_2 \\
\Delta_5 &= c_{44}\Delta_1 - \epsilon_0 c\Delta_2.
\end{aligned}
\tag{3.7}
$$

It is clear that $\Delta_i (i = 1,2,4,5)$ are linear and Δ_3 is a quadratic in q.

From Eqns. (3.2–3.7) one obtains the representations

$$
\begin{aligned}
\bar{u} &= -q_2\nabla^2\Delta_1\bar{\Phi}_1 + \Delta_3\nabla\nabla.\bar{\Phi}_1 + d_{44}\Delta_1\nabla^2\bar{\Phi}_2 - \Delta_4\nabla\nabla.\bar{\Phi}_2 - d\nabla\Psi \\
\bar{P} &= d_{44}\Delta_1\nabla^4\bar{\Phi}_1 - \Delta_4\nabla^2\nabla\nabla.\bar{\Phi}_1 - c_{44}\Delta_1\nabla^2\bar{\Phi}_2 + \Delta_5\nabla\nabla.\bar{\Phi}_2 + c\nabla\Psi \\
\phi &= d\Delta_2\nabla^2\nabla.\bar{\Phi}_1 - c\Delta_2\nabla.\bar{\Phi}_2 + [(bc-d^2)\nabla^2 - ac]\Psi
\end{aligned}
\tag{3.8}
$$

where $\bar{\Phi}_1$, $\bar{\Phi}_2$ and Ψ satisfy the equations

$$
\begin{aligned}
\Delta_1\Delta_2\nabla^4\bar{\Phi}_1 &= -\bar{f} \\
\Delta_1\Delta_2\nabla^2\bar{\Phi}_2 &= -\bar{E}_0 \\
\Delta_1\nabla^2\Psi &= -\rho_c.
\end{aligned}
\tag{3.9}
$$

4. MATRICES OF FUNDAMENTAL SOLUTIONS

(a) *Concentrated force*

Let $4\pi\bar{e}_p$ be a concentrated force applied at the point $y(y_1,y_2,y_3)$ of an unbounded isotropic elastic dielectric. Here $\bar{e}_p$ is a unit vector along the x_p-direction of the Cartesian coordinate system. Then

$$
\bar{f}^{(p)}(x) = 4\pi\bar{e}_p\delta(y-x), \quad \bar{E}_0 = 0, \quad \phi = 0 \tag{4.1}
$$

By standard methods[8], the solution of the system (3.9) is, given by

$$
\begin{aligned}
\bar{\Phi}_1^{(p)} &= -\frac{\bar{e}_p}{\delta_1^2\delta_2^2}\left\{-\frac{r}{2\alpha_1^2\alpha_2^2} + \frac{\alpha_1^2+\alpha_2^2}{\alpha_1^4\alpha_2^4}\frac{1}{r} + \frac{1}{\alpha_1^2-\alpha_2^2}\left(\frac{\alpha_1^{-4}e^{i\alpha_1 r} - \alpha_2^{-4}e^{i\alpha_2 r}}{r}\right)\right\} \\
\bar{\Phi}_2 &= \bar{0}, \quad \bar{\Psi} = 0
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
\alpha_1^2 &= c(1+a\epsilon_0)/\delta_1^2, \qquad \alpha_2^2 = \frac{ac_{44}}{\delta_2^2} \\
\delta_1^2 &= \epsilon_0(d^2-bc), \qquad \delta_2^2 = d_{44}^2 - c_{44}(b_{44}+b_{77}).
\end{aligned}
\tag{4.3}
$$

Let ${}_F\bar{\Gamma}^p(x,y)$, ${}_F\bar{\Omega}^p(x,y)$ and ${}_F\chi^p$ denote, respectively, the displacement vector, the polarization vector and the potential of the Maxwell field corresponding to the above concentrated force.

Substituting (4.2) into (3.8), one obtains the fundamental matrix solutions,

$$
\begin{aligned}
{}_F\Gamma_j^p &= \frac{1}{\alpha_2^2\delta_2^2}\frac{[a - A_2 e^{i\alpha_2 r}]}{r}\delta_{jp} + \frac{\partial^2}{\partial x_j \partial x_p}\left\{\left(\frac{1+a\epsilon_0}{\alpha_1^2\delta_1^2} - \frac{a}{\alpha_2^2\delta_2^2}\right)\frac{r}{2}\right. \\
&\quad \left. + \frac{\delta_1^{-2}\alpha_1^{-4}A_1(e^{i\alpha_1 r}-1) - \delta_2^{-2}\alpha_2^{-4}A_2(e^{i\alpha_2 r}-1)}{r}\right\}, \\
{}_F\Omega_j^p &= \frac{d_{44}}{\delta_2^2}\frac{e^{i\alpha_2 r}}{r}\delta_{pj} - \frac{\partial^2}{\partial x_j \partial x_p}\left\{-\frac{d_{44}}{\alpha_2^2\delta_2^2}\frac{e^{i\alpha_2 r}-1}{r} + \frac{\epsilon_0 d}{\alpha_1^2\delta_1^2}\frac{e^{i\alpha_1 r}-1}{r}\right\}, \\
{}_F\chi_p &= -\frac{d}{\alpha_1^2\delta_1^2}\frac{\partial}{\partial x_p}\left[\frac{e^{i\alpha_1 r}-1}{r}\right]
\end{aligned}
\tag{4.4}
$$

where $A_1 = 1 + a\epsilon_0 + \epsilon_0(b_{12}+2b_{44})\alpha_1^2$, $A_2 = a + (b_{44}+b_{77})\alpha_2^2$.

(b) *Concentrated electric force*

Let $4\pi\bar{e}_p$ be a concentrated electric force acting at the point y. Then

$$
\bar{f} = \bar{0}, \quad \bar{E}_0^{(p)} = 4\pi\bar{e}_p\,\delta(y-x), \quad \rho_c = 0. \tag{4.5}
$$

The solution of the system (3.9) is given by

$$
\begin{aligned}
\bar{\Phi}_1 &= 0, \; \bar{\Phi}_2^{(p)} = \frac{1}{\delta_1^2\delta_2^2}\left\{\frac{(\alpha_1\alpha_2)^{-2}}{r} + \frac{\alpha_1^{-2}e^{i\alpha_1 r} - \alpha_2^{-2}e^{i\alpha_2 r}}{(\alpha_1^2-\alpha_2^2)r}\right\}, \\
\bar{\Psi} &= 0.
\end{aligned}
\tag{4.6}
$$

Substituting from (4.6) into (3.8), one obtains the fundamental matrix solutions

$$
\begin{aligned}
{}_H\Gamma_j^p &= \frac{d_{44}}{\delta_2^2}\frac{e^{i\alpha_2 r}}{r}\delta_{pj} - \frac{\partial^2}{\partial x_p \partial x_j}\left[\frac{\epsilon_0 d}{\alpha_1^2\delta_1^2}\frac{e^{i\alpha_1 r}-1}{r} - \frac{d_{44}}{\alpha_2^2\delta_2^2}\frac{e^{i\alpha_2 r}-1}{r}\right] \\
{}_H\Omega_j^p &= -\frac{c_{44}}{\delta_2^2}\frac{e^{i\alpha_2 r}}{r}\delta_{pj} + \frac{\partial^2}{\partial x_p \partial x_j}\left[\frac{\epsilon_0 c}{\alpha_1^2\delta_1^2}\frac{e^{i\alpha_1 r}-1}{r} - \frac{c_{44}}{\alpha_2^2\delta_2^2}\frac{e^{i\alpha_2 r}-1}{r}\right] \\
{}_H\chi_p &= \frac{c}{\alpha_1^2\delta_1^2}\frac{\partial}{\partial x_p}\left[\frac{e^{i\alpha_1 r}-1}{r}\right].
\end{aligned}
\tag{4.7}
$$

(c) *Concentrated charge*

Let $4\pi\delta(x-y)$ be the concentrated charge acting at the point y. Then

$$
\bar{f} = \bar{0}, \quad \bar{E} = \bar{0}, \quad \rho_c = 4\pi\delta(x-y). \tag{4.8}
$$

The solution of the system (3.9), in this case, is given by

$$
\bar{\Phi}_1 = \bar{0}, \; \bar{\Phi}_2 = 0, \; \bar{\Psi} = -\frac{1}{\alpha_1^2}\frac{[e^{i\alpha_1 r}-1]}{r}. \tag{4.9}
$$

Substituting from (4.9) into (3.8), one obtains the singular solutions

$$
\begin{aligned}
{}_c\Gamma_j &= \frac{d}{\alpha_1^2}\frac{\partial}{\partial x_j}\left[\frac{e^{i\alpha_1 r}-1}{r}\right],\\
{}_c\Omega_j &= -\frac{c}{\alpha_1^2}\frac{\partial}{\partial x_j}\left[\frac{e^{i\alpha_1 r}-1}{r}\right]\\
{}_c\chi &= -\frac{\delta_1^2}{\epsilon_0}\frac{e^{i\alpha_1 r}}{r}+\frac{ac}{\alpha_1^2}\left[\frac{e^{i\alpha_1 r}-1}{r}\right].
\end{aligned}
\tag{4.10}
$$

5. DIELECTRIC POTENTIALS

As in ordinary potential theory and the potentials of Kupradze[5] in elastokinetics, we introduce

Potentials of single layer:

$$
\begin{aligned}
U_j(x;\bar{\Psi}^1,\bar{\Psi}^2,\nu) &= \frac{1}{2\pi}\int_s [{}_F\Gamma_j^p(x,y)\Psi_p^1(y)+{}_F\Omega_j^p(x,y)\Psi_p^2(y)+{}_F\chi_j(x,y)\nu(y)]\,\mathrm{d}s_y\\
W_j(x;\bar{\Psi}^1,\bar{\Psi}^2,\nu) &= \frac{1}{2\pi}\int_s [{}_H\Gamma_j^p(x,y)\Psi_p^1(y)+{}_H\Omega_j^p(x,y)\Psi_p^2(y)+{}_H\chi_j(x,y)\nu(y)]\,\mathrm{d}s_y\\
S(x;\bar{\Psi}^1,\bar{\Psi}^2,\nu) &= \frac{1}{2\pi}\int_s [{}_c\Gamma_j(x,y)\Psi_j^1(y)+{}_c\Omega_j(x,y)\Psi_j^2(y)+{}_c\chi(x,y)\nu(y)]\,\mathrm{d}s_y.
\end{aligned}
\tag{5.1}
$$

Potentials of double layer:

$$
\begin{aligned}
\tilde{U}_p(x;\bar{K}^1,\bar{K}^2,\mu) &= \frac{1}{2\pi}\int_s [R_j({}_F\Gamma^p,{}_F\Omega^p)K_j^1(y)+M_j({}_F\Gamma^p,{}_F\Omega^p)K_j^2(y)+Y({}_F\Omega^p,{}_F\chi)\mu(y)]\,\mathrm{d}s_y\\
\tilde{W}_p(x;\bar{K}^1,\bar{K}^2,\mu) &= \frac{1}{2\pi}\int_s [R_j({}_H\Gamma^p,{}_H\Omega^p)K_j^1(y)+M_j({}_H\Gamma^p,{}_H\Omega^p)K_j^2(y)+Y({}_H\Omega^p,{}_H\chi)\mu(y)]\,\mathrm{d}s_y\\
\tilde{S}(x;\bar{K}^1,\bar{K}^2,\mu) &= \frac{1}{2\pi}\int_s [R_j({}_c\Gamma,{}_c\Omega)K_j^1(y)+M_j({}_c\Gamma,{}_c\Omega)K_j^2(y)+Y({}_c\Omega,{}_c\chi)\mu(y)]\,\mathrm{d}s_y.
\end{aligned}
\tag{5.2}
$$

Throughout the rest of the paper $H(\gamma)$ and C stand, respectively, for Hölder class with exponent γ and a class of continuous functions. A Lyapunov surface is defined as a closed surface S with a continuously turning tangent plane; E denotes the entire space.

Now the following theorems concerning the continuity of the above potentials are proved.

Theorem 1. For density functions $\bar{\Psi}^1(y),\bar{\Psi}^2(y)$ and $\nu(y)\in C$, the single layer potentials $U(x;\bar{\Psi}^1,\bar{\Psi}^2,\nu)$, $W(x;\bar{\Psi}^1,\bar{\Psi}^2,\nu)$ and $S(x;\bar{\Psi}^1,\bar{\Psi}^2,\nu)$ are continuous for $x\in E$.

It is clear that $U(x;\bar{\Psi}^1,\bar{\Psi}^2,\nu)$, $W(x;\bar{\Psi}^1,\bar{\Psi}^2,\nu)$ and $S(x;\bar{\Psi}^1,\bar{\Psi}^2,\nu)$, as given by Eqn (5.1) exist and are continuous for all points in space except possibly points on the surface S.

Expanding exponentials in appropriate series in the kernels ${}_F\Gamma^p$, ${}_F\Omega^p$, ${}_F\chi_p$; ${}_H\Gamma^p$, ${}_H\Omega^p$, ${}_H\chi_p$; ${}_c\Gamma$, ${}_c\Omega$, ${}_c\chi$, one obtains

$$
{}_F\Gamma_j^p = -\frac{1}{2r}\left[\frac{\epsilon_0 b}{\delta_1^2}+\frac{b_{44}+b_{77}}{\delta_2^2}\right]\delta_{jp}+\frac{1}{2r}\left[\frac{\epsilon_0 b}{\delta_1^2}-\frac{b_{44}+b_{77}}{\delta_2^2}\right]\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_p}+0(r^n).\ n\geq 0
$$

$$
\begin{aligned}
{}_F\Omega_j^p &= \frac{1}{2r}\left[\frac{\epsilon_0 d}{\delta_1^2}+\frac{d_{44}}{\delta_2^2}\right]\delta_{jp} - \frac{1}{2r}\left[\frac{\epsilon_0 d}{\delta_1^2}-\frac{d_{44}}{\delta_2^2}\right]\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_p} + 0(r^n),\ n \geq 0 \\
{}_F\chi_i &= \frac{d}{2\delta_1^2}\frac{\partial r}{\partial x_j} + 0(r^n),\ n \geq 0 \\
{}_H\Gamma_j^p &= \frac{1}{2r}\left[\frac{\epsilon_0 d}{\delta_1^2}+\frac{d_{44}}{d_2^2}\right]\delta_{jp} - \frac{1}{2r}\left[\frac{\epsilon_0 d}{\delta_1^2}-\frac{d_{44}}{\delta_2^2}\right]\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_p} + 0(r^n),\ n \geqslant 0 \\
{}_H\Omega_j^p &= -\frac{1}{2r}\left[\frac{\epsilon_0 c}{\delta_1^2}+\frac{c_{44}}{\delta_2^2}\right]\delta_{jp} + \frac{1}{2r}\left[\frac{\epsilon_0 c}{\delta_1^2}-\frac{c_{44}}{\delta_2^2}\right]\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_p} + 0(r^n),\ n \geq 0 \\
{}_H\chi_i &= -\frac{c}{2\delta_1^2}\frac{\partial r}{\partial x_j} + 0(r^n),\ n \geq 0 \\
{}_c\Gamma_j &= -\frac{d}{2}\frac{\partial r}{\partial x_j} + 0(r^n),\ n \geq 0 \\
{}_c\Omega_j &= \frac{c}{2}\frac{\partial r}{\partial x_j} + 0(r^n),\ n \geq 0 \\
{}_c\chi &= -\frac{\delta_1^2}{\epsilon_0}\frac{1}{r} + 0(r^n),\ n \geq 0
\end{aligned}
\tag{5.3}
$$

and hence, it is obvious that U, W and S exist and are continuous for points on S as well.

Theorem 2. For density functions $\bar{K}^1(y)$, $\bar{K}^2(y)$, $\mu(y) \in H(\gamma)$ and S a Lyapunov surface, the double layer potentials $\tilde{U}_p(x;\bar{K}^1,\bar{K}^2,\mu)$, $\tilde{W}_p(x;\bar{K}^1,\bar{K}^2,\mu)$ and $\tilde{S}(x;\bar{K}^1,\bar{K}^2,\mu)$ tend to the following finite limits as x tends to $x_0 \in S$ from inside and from outside,

$$
\begin{aligned}
[\tilde{U}_p(x;\bar{K}^1,\bar{K}^2,\mu)]_{i,e} &= \pm K_p^1 + \frac{1}{2\pi}\int_s [R_j({}_F\Gamma^p,{}_F\Omega^p)K_j^1(y) + M_j({}_F\Gamma^p,{}_F\Omega^p)K_j^2(y) \\
&\quad + Y({}_F\Omega^p,{}_F\chi)\mu(y)]\,\mathrm{d}s_y \\
[\tilde{W}_p(x;\bar{K}^1,\bar{K}^2,\mu)]_{i,e} &= \pm K_p^2 + \frac{1}{2\pi}\int_s [R_j({}_H\Gamma^p,{}_H\Omega^p)K_j^1(y) + M_j({}_H\Gamma^p,{}_H\Omega^p)K_j^2(y) \\
&\quad + Y({}_H\Omega^p,{}_H\chi)\mu(y)]\,\mathrm{d}s_y \\
[\tilde{S}(x;\bar{K}^1,\bar{K}^2,\mu)]_{i,e} &= \pm \delta_1^2\mu(y) + \frac{1}{2\pi}\int_s [R_j({}_c\Gamma,{}_c\Omega)K_j^1(y) + M_j({}_c\Gamma,{}_c\Omega)K_j^2(y) \\
&\quad + Y({}_c\Omega,{}_c\chi)\mu(y)]\,\mathrm{d}s_y
\end{aligned}
\tag{5.4}
$$

where $i,+$ and $e,-$ correspond respectively to the limits from interior and exterior of S and the integrals are imagined in the sense of Cauchy's Principal values.

The expressions for the double layer potentials can be written in the forms

$$
\begin{aligned}
\tilde{U}_p(x;\bar{K}^1,\bar{K}^2,\mu) &= \frac{1}{2}\int_s \{R_j({}_F\Gamma^p,{}_F\Omega^p)[K_j^1(y)-K_j^1(x_0)] + M_j({}_F\Gamma^p,{}_F\Omega^p)[K_j^2(y)-K_j^2(x_0)]\}\,\mathrm{d}s_y \\
&\quad + \frac{K_j^1(x_0)}{2\pi}\int_s R_j({}_F\Gamma^p,{}_F\Omega^p)\,\mathrm{d}s_y + \frac{K_j^2(x_0)}{2\pi}\int_s M_j({}_F\Gamma^p,{}_F\Omega^p)\,\mathrm{d}s_y \\
&\quad + \frac{1}{2\pi}\int_s Y({}_F\Omega^p,{}_F\chi)\mu(y)\,\mathrm{d}s_y
\end{aligned}
\tag{5.5}
$$

$$\tilde{W}_p(x;\bar{K}^1,\bar{K}^2,\mu)=\frac{1}{2\pi}\int_s \{R_j({}_H\Gamma^p,{}_H\Omega^p)[K_j^1(y)-K_j^1(x_0)]+M_j({}_H\Gamma^p,{}_H\Omega^p)[K_j^2(y)-K_j^2(x_0)]\}\,\mathrm{d}s_y$$
$$+\frac{K_j^1(x_0)}{2\pi}\int_s R_j({}_H\Gamma^p,{}_H\Omega^p)\,\mathrm{d}s_y+\frac{K_j^2(x_0)}{2\pi}\int_s M_j({}_F\Gamma^p,{}_F\Omega^p)\,\mathrm{d}s_y$$
$$+\frac{1}{2\pi}\int_s Y({}_H\Omega^p,{}_H\chi)\mu(y)\,\mathrm{d}s_y \tag{5.6}$$

$$S(x;\bar{K}^1,\bar{K}^2,\mu)=\frac{1}{2\pi}\int_s [R_j({}_c\Gamma,{}_c\Omega)K_j^1(y)+M_j({}_c\Gamma,{}_c\Omega)K_j^2]\,\mathrm{d}s_y+\frac{1}{2\pi}\int_s Y({}_c\Omega,{}_c\chi)$$
$$\times[\mu(y)-\mu(x_0)]\,\mathrm{d}s_y+\frac{\mu(x_0)}{2\pi}\int_s Y({}_c\Omega,{}_c\chi)\,\mathrm{d}s_y \tag{5.7}$$

where the expressions for $R({}_x\Gamma,{}_x\Omega)$, $M({}_x\Gamma,{}_x\Omega)$, $Y({}_x\Omega,{}_x\chi)$ $(x=F,H,c)$, after the exponentials are expanded in the appropriate series, are given by

$$R_j({}_F\Gamma^p,{}_F\Omega^p)=\frac{\eta}{r^2}f_{jp}(r,\bar{n})-\frac{3}{r^2}\frac{\partial r}{\partial n}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_p}+0(r^{n-1}),\ n\geq 0$$
$$M_j({}_F\Gamma^p,{}_F\Omega^p)=-\frac{\beta}{r^2}f_{jp}(r,\bar{n})-\frac{2b_{77}d_{44}}{\delta_2{}^2r^2}\left(n_j\frac{\partial r}{\partial x_p}-n_p\frac{\partial r}{\partial x_j}\right)+0(r^{n-1}),\ n\geq 0$$
$$Y({}_F\Omega^p,{}_F\chi)=0(r^{n-1}),\ n\geq 0$$
$$R_j({}_H\Gamma^p,{}_H\Omega^p)=\frac{\xi}{r^2}f_{jp}(r,\bar{n})+0(r^{n-1}),\ n\geq 0$$
$$M_j({}_H\Gamma^p,{}_H\Omega^p)=\frac{\zeta}{r^2}f_{jp}(r,\bar{n})+\frac{2b_{77}c_{44}}{\delta_2{}^2r^2}\left(n_j\frac{\partial r}{\partial x_p}-n_p\frac{\partial r}{\partial x_j}\right)-\frac{3}{r^2}\frac{\partial r}{\partial n}\frac{\partial r}{\partial x_p}\frac{\partial r}{\partial x_j}+0(r^{n-1}),\ n\geq 0$$
$$Y({}_H\Omega^p,{}_H\chi)=0(r^{n-1}),\ n\geq 0$$
$$R({}_c\Gamma,{}_c\Omega)=0(r^{n-1}),\ M({}_c\Gamma,{}_c\Omega)=0(r^{n-1}),\ n\geq 0$$
$$Y({}_c\Omega,{}_c\chi)=-\frac{\delta_1{}^2}{r^2}\frac{\partial r}{\partial n}+0(r^{n-1}),\ n\geq 0 \tag{5.8}$$

and where

$$f_{jp}(r,\bar{n})=-n_j\frac{\partial r}{\partial x_p}+n_p\frac{\partial r}{\partial x_j}+\delta_{jp}\frac{\partial r}{\partial n}-3\frac{\partial r}{\partial n}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_p}$$
$$\xi=\frac{\epsilon_0}{\delta_1{}^2}(c_{12}d_{44}-d_{12}c_{44})$$
$$\eta=\frac{\epsilon_0}{\delta_1{}^2}[c_{44}(b_{12}+2b_{44})-d_{44}(d_{12}+2d_{44})] \tag{5.9}$$
$$\beta=+\frac{\epsilon_0}{\delta_1{}^2}(d_{12}b_{44}-b_{12}d_{44})+\frac{b_{77}c_{44}}{\delta_2{}^2}$$
$$\zeta=\frac{\epsilon_0}{\delta_1{}^2}[b_{44}(c_{12}+2c_{44})-d_{44}(d_{12}+2d_{44})]+\frac{b_{77}c_{44}}{\delta_2{}^2}.$$

The first and the fourth surface integrals in (5.5 and 5.6) and the second and the first surface integrals in (5.7) are continuous as x tends to $x_0 \in S$, since $\bar{K}^1(y)$, $\bar{K}^2(y)$ and $\mu(y) \in H(\gamma)$ and the quantities $Y({}_F\Omega^p,{}_F\chi)$, $Y({}_H\Omega^p,{}_H\chi)$, $R_j({}_c\Gamma,{}_c\Omega)$ and $M_j({}_c\Gamma,{}_c\Omega)$ have a singularity of order $1/r$. The limit of the second and the third surface integrals (5.5 and 5.6) and the third surface integral in (5.7) are given by

$$\lim_{x\to x_0}\int R^{(y)}({}_X\Gamma^p,{}_X\Omega^p)\,ds_y = \lim_{\epsilon\to 0}\int_{s-\sigma(x_0;\epsilon)} R^{(y)}({}_X\Gamma^p,{}_X\Omega^p)\,ds_y + \lim_{\epsilon\to 0}\int_{\sigma(x_0;\epsilon)} R^{(y)}({}_X\Gamma^p,{}_X\Omega^p)\,ds_y$$

$$\lim_{x\to x_0}\int_s M^{(y)}({}_X\Gamma^p,{}_X\Omega^p)\,ds_y = \lim_{\epsilon\to 0}\int_{s-\sigma(x_0;\epsilon)} M^{(y)}({}_X\Gamma^p,{}_X\Omega^p)\,ds_y + \lim_{\epsilon\to 0}\int_{\sigma(x_0;\epsilon)} M^{(y)}({}_X\Gamma^p,{}_X\Omega^p)\,ds_y$$

where $X = F, H$ and

$$\lim_{x\to x_0}\int_s Y^{(y)}({}_c\Omega^p,{}_c\chi)\,ds_y = \lim_{\epsilon\to 0}\int_{s-\sigma(x_0;\epsilon)} Y^{(y)}({}_c\Omega^p,{}_c\chi)\,ds_y + \lim_{\epsilon\to 0}\int_{\sigma(x_0;\epsilon)} Y^{(y)}({}_c\Omega^p,{}_c\chi)\,ds_y \tag{5.10}$$

where $\sigma(x_0;\epsilon)$ is the surface of a sphere with centre at x_0 and radius ϵ which indents or is superimposed on the domain D bounded by surface S, accordingly as the point x approaches the point $x_0 \in S$ from the interior or the exterior of the domain and thus isolates the point of singularity x_0.

For a point $y \in \sigma(x_0;\epsilon)$, we take the direction of the outward normal at y to $s \mp \sigma(x_0;\epsilon)$ as positive and

$$\left[\frac{\partial^{(y)}}{\partial n}\right]_{i,e} = \mp\frac{\partial}{\partial r}, \quad [n_j^y]_{i,e} = \mp\frac{\partial r}{\partial y_j}. \tag{5.11}$$

Making use of the above convention, Eqns (5.8) approach the following finite limits at $y \in \sigma(x_0;\epsilon)$.

$$\begin{aligned}
[R_j({}_F\Gamma^p,{}_F\Omega^p)]_{i,e} &= \mp\frac{1}{r^2}\left[\eta\delta_{jp} - 3(1+\eta)\frac{\partial r}{\partial y_j}\frac{\partial r}{\partial y_p}\right] + 0(r^{n-1})\\
[M_j({}_F\Gamma^p,{}_F\Omega^p)]_{i,e} &= \mp\frac{(-\beta)}{r^2}\left[\delta_{jp} - 3\frac{\partial r}{\partial y_j}\frac{\partial r}{\partial y_p}\right] + 0(r^{n-1})\\
[R_j({}_H\Gamma^p,{}_H\Omega^p)]_{i,e} &= \mp\frac{\xi}{r^2}\left[\delta_{jp} - 3\frac{\partial r}{\partial y_j}\frac{\partial r}{\partial y_p}\right] + 0(r^{n-1})\\
[M_j({}_H\Gamma^p,{}_H\Omega^p)]_{i,e} &= \mp\frac{1}{r^2}\left[\zeta\delta_{jp} - 3(1+\zeta)\frac{\partial r}{\partial y_j}\frac{\partial r}{\partial y_p}\right] + 0(r^{n-1})\\
[Y({}_c\Omega^p,{}_c\chi)]_{i,e} &= \mp\frac{\delta_1^2}{r^2}.
\end{aligned} \tag{5.12}$$

Now, making use of (5.11) and the following results

$$\lim_{\epsilon\to 0}\int_{\sigma(x_0;\epsilon)}\frac{1}{r^2}\left(\frac{\partial r}{\partial y_j}\frac{\partial r}{\partial y_k}, 1\right) ds_y = 2\pi\left(\frac{1}{3}\delta_{jk}, 1\right) \tag{5.13}$$

in (5.10), one finds that

$$\lim_{x\to x_0}\left[\int_S R^{(y)}({}_F\Gamma,{}_F\Omega)\,ds_y\right]_{i,e} = \int_S R^{(y)}({}_F\Gamma,{}_F\Omega)(x_0,y)\,ds_y \pm 2\pi$$

$$\lim_{x\to x_0}\left[\int_S M^{(y)}({}_F\Gamma,{}_F\Omega)\,ds_y\right]_{i,e} = \int_S M^{(y)}({}_F\Gamma,{}_F\Omega)(x_0,y)\,ds_y$$

$$\lim_{x\to x_0}\left[\int_S R^{(y)}({}_H\Gamma,{}_H\Omega)\,ds_y\right]_{i,e} = \int_S R^{(y)}({}_H\Gamma,{}_H\Omega)(x_0,y)\,ds_y$$

$$\lim_{x\to x_0}\left[\int_S M^{(y)}({}_H\Gamma,{}_H\Omega)\,ds_y\right]_{i,e} = \int_S M^{(y)}({}_H\Gamma,{}_H\Omega)(x_0,y)\,ds_y \pm 2\pi$$

$$\lim_{x\to x_0}\left[\int_S Y^{(y)}({}_c\Omega,{}_c\chi)\,ds_y\right]_{i,e} = \int_S Y^{(y)}({}_c\Omega,{}_c\chi)\,ds_y \pm 2\pi\delta_i^2. \tag{5.14}$$

From (5.14) and the limits of both sides of (5.5), one obtains the results (5.4).

Theorem 3. For density functions $\bar{\Psi}^1(y)$, $\bar{\Psi}^2(y)$, $\nu(y)\in H(\gamma)$ and S a Lyapunov surface, the application of R, M and Y operators on potentials of single layer results in functions which tend to the following finite limits as x approaches $x_0 \in S$ from inside and outside.

$$\begin{aligned}
[R(U,W)]_{i,e} &= \pm\bar{\Psi}^1(x_0)+\frac{1}{2\pi}\int_S R^{(x_0)}[{}_F\Gamma(x_0,y)\bar{\Psi}^1(y)+{}_F\Omega(x_0,y)\bar{\Psi}^2(y)\\
&\quad +{}_F\chi(x_0,y)\nu(y),{}_H\Gamma(x_0,y)\bar{\Psi}^1(y)+{}_H\Omega(x_0,y)\bar{\Psi}^2(y)+{}_H\chi(x_0,y)\nu(y)]\,ds_y\\
[M(U,W)]_{i,e} &= \pm\bar{\Psi}^2(x_0)+\frac{1}{2\pi}\int_S M^{(x_0)}[{}_F\Gamma(x_0,y)\bar{\Psi}^1(y)+{}_F\Omega(x_0,y)\bar{\Psi}^2(y)\\
&\quad +{}_F\chi(x_0,y)\nu(y),{}_H\Gamma(x_0,y)\bar{\Psi}^1(y)+{}_H\Omega(x_0,y)\bar{\Psi}^2(y)+{}_H\chi(x_0,y)\nu(y)]\,ds_y\\
[Y(W,S)]_{i,e} &= \pm\delta_i^2\nu(y)+\frac{1}{2\pi}\int_S Y^{(x_0)}[{}_H\Gamma(x_0,y)\bar{\Psi}^1(y)+{}_H\Omega(x_0,y)\bar{\Psi}^2(y)\\
&\quad +{}_H\chi(x_0,y)\nu(y),{}_c\Gamma(x_0,y)\bar{\Psi}^1(y)+{}_c\Omega(x_0,y)\bar{\Psi}^2(y)+{}_c\chi(x_0,y)\nu(y)]\,ds_y
\end{aligned} \tag{5.15}$$

where the integrals are to be understood in the sense of Cauchy's Principal values.

The expansions of exponential terms in singular kernel matrices $R({}_x\Gamma,{}_x\Omega,{}_x\chi)$, $M({}_x\Gamma,{}_x\Omega,{}_x\chi)$, $Y({}_x\Gamma,{}_x\Omega,{}_x\chi)X = F, H, C$ have been given in Theorem 2. Keeping in mind that for $x \neq y \in S$, the product of linear operators at the points x and y, operating on a function is commutative, the results (5.15) can be proved following the procedure of Theorem 2.

6. SINGULAR INTEGRAL EQUATIONS

For the first interior and exterior boundary value problem of linear elastic dielectrics, we seek the solution of (2.6) as double layer potentials (5.2) such that for $x \in S$, the displacement vector, the polarization vector and the potential of Maxwell field are prescribed functions $\bar{U}_{i,e}$, $\bar{p}_{i,e}$ and $\bar{\Phi}_{i,e} \in H(\gamma)$ respectively.

For the second interior and exterior boundary value problem, we seek solution of (2.6) as single layer potentials (5.1) such that for $x \in S$, the stress vector $n_i T_{ij}$, the electric force vector $n_i E_{ij}$ and $n_i[-\epsilon_0\|\phi_i\| + P_i]$ are prescribed functions $\bar{k}(x)_{i,e}$, $\bar{S}(x)_{i,e}$ and $\theta(x)_{i,e} \in H(\gamma)$ respectively.

Using the discontinuity theorems of surface potentials proved in Section 5, the singular integral equations for the fundamental boundary value problems, involving unknown density functions $\bar{\Psi}^1(\Psi_1^1,\Psi_2^1,\Psi_3^1)$, $\bar{\Psi}^2(\Psi_1^2,\Psi_2^2,\Psi_3^2)$,$\nu$ and $\bar{K}^1(K_1^1,K_2^1,K_3^1)$, $\bar{K}^2(K_1^2,K_2^2,K_3^2)$,$\mu$ may be written in the following form:

$$K_p^1(x)\pm\frac{1}{2\pi}\int_s\left[R_j^{(y)}({}_F\Gamma^p,{}_F\Omega^p)K_j^1(y)+M_j^{(y)}({}_F\Gamma^p,{}_F\Omega^p)K_j^2(y)+Y^{(y)}({}_F\Omega^p,{}_F\chi)\mu(y)\right]ds_y=[\tilde{U}_p]_{i,e}$$

$$K_p^2(x)\pm\frac{1}{2\pi}\int_s[R_j^{(y)}({}_H\Gamma^p,{}_H\Omega^p)K_j^1(y)+M_j^{(y)}({}_H\Gamma^p,{}_H\Omega^p)K_j^2(y)+Y^{(y)}({}_H\Omega^p,{}_H\chi)\mu(y)]\,ds_y=[\tilde{P}_p]_{i,e}$$

$$\delta_1^2\mu(x)\pm\frac{1}{2\pi}\int_s R^{(y)}({}_c\Gamma,{}_c\Omega)K_j^1(y)+M^{(y)}({}_c\Gamma,{}_c\Omega)K_j^2(y)+Y^{(y)}({}_c\Omega,{}_c\chi)\mu(y)\,ds_y=\pm[\tilde{\phi}]_{i,e}\qquad(6.1)$$

and

$$\bar{\Psi}^1(x)\pm\frac{1}{2\pi}\int_s R^{(y)}[({}_F\Gamma(x,y)\bar{\Psi}^1(y)+{}_F\Omega(x,y)\bar{\Psi}^2(y)+{}_F\chi(x,y)\nu(y),{}_H\Gamma(x,y)\bar{\Psi}^1(y)+{}_H\Omega(x,y)\bar{\Psi}^2(y)+{}_H\chi(x,y)\nu(y))]\,ds_y=\pm[\bar{k}(x)]_{i,e}\qquad(6.2)$$

$$\bar{\Psi}^2(x)\pm\frac{1}{2\pi}\int_s M^{(y)}[{}_F\Gamma(x,y)\bar{\Psi}^1(y)+{}_F\Omega(x,y)\bar{\Psi}^2(y)+{}_F\chi(x,y)\nu(y),{}_H\Gamma(x,y)\bar{\Psi}^1(y)+{}_H\Omega(x,y)\bar{\Psi}^2(y)+{}_H\chi(x,y)\nu(y)]\,ds_y=\pm[S(x)]_{i,e}$$

$$\delta_1^2\nu(x)\pm\frac{1}{2\pi}\int_s Y^{(y)}[{}_H\Gamma(x,y)\bar{\Psi}^1(y)+{}_H\Omega(x,y)\bar{\Psi}^2(y)+{}_H\chi(x,y)\nu(y),{}_c\Gamma(x,y)\bar{\Psi}^1+{}_c\Omega(x,y)\bar{\Psi}^2(x)+{}_c\chi(x,y)\nu(y)]\,dsy=\pm(\theta(x)]_{i,e}.$$

Further, we investigate the symbolic determinant [7] of the system of seven equations (6.1) in the seven unknown density functions K_1^1, K_2^1, K_3^1, K_1^2, K_2^2, K_3^2, $\mu(y)$. Keeping in mind that the particular choice of coordinates will not alter the singularities of the involved kernels, we introduce local coordinates at each point $x\in S$ directing x_3-axis along the outside normal to S and the x_1, x_2 axes in the tangent plane to S. Expanding the kernels and noting that in the local coordinate system $n_1=0$, $n_2=0$, $n_3=1$, we represent the system (6.1) for the first interior boundary value problem in the form:

$$K_1^1(x)+\frac{1}{2\pi}\int_s\frac{\cos\theta}{r^2}[-\eta K_3^1(y)+\beta_1K_3^2(y)]\,ds_y+L_1(\bar{K}^1,\bar{K}^2,\mu)=[\tilde{U}_1]_i$$

$$K_2^1(x)+\frac{1}{2\pi}\int_s\frac{\sin\theta}{r^2}[-\eta K_3^1(y)+\beta_1K_3^2(y)]\,ds_y+L_2(\bar{K}^1,\bar{K}^2,\mu)=[\tilde{U}_2]_i$$

$$K_3^1(x)+\frac{1}{2\pi}\int_s\frac{1}{r^2}[(\eta K_1^1-\beta_1K_1^2)\cos\theta+(\eta K_2^1-\beta_1K_2^2)\sin\theta]\,ds_y+L_3(\bar{K}^1,\bar{K}^2,\mu)=[\tilde{U}_3]_i$$

$$K_1^2(x)+\frac{1}{2\pi}\int_s\frac{\cos\theta}{r^2}[-\xi K_3^1+\zeta_1K_3^2]\,ds_y+L_4(\bar{K}^1,\bar{K}^2,\mu)=[\tilde{p}_1]_i$$

$$K_2^2(x)+\frac{1}{2\pi}\int_s\frac{\sin\theta}{r^2}[-\xi K_3^1+\zeta_1K_3^2]\,ds_y+L_5(\bar{K}^1,\bar{K}^2,\mu)=[\tilde{p}_2]_i$$

$$K_3^2(x)+\frac{1}{2\pi}\int_s\frac{1}{r^2}[(\xi K_1^1-\zeta_1K_1^2)\cos\theta+(\xi K_2^1-\zeta_1K_2^2)\sin\theta]\,ds_y+L_6(\bar{K}_1,\bar{K}_2,\mu)=[\tilde{p}_3]_i$$

$$\delta_1^2\mu(x)+L_7(\bar{K}_1,\bar{K}_2,\mu)=[\tilde{\phi}]_i\qquad(6.3)$$

where

$$\cos\theta = \frac{\partial r}{\partial y_1},\ \sin\theta = \frac{\partial r}{\partial y_2},\ (\beta_1,\zeta_1) = -(\beta,\zeta) + \frac{2b_{77}}{\delta_2{}^2}(d_{44},c_{44})$$

and L_k $(k = 1, 2, \ldots, 7)$ are integral operators with weak singularities.

The symbolic determinant of the system (6.3) is given by

$$\Delta = \begin{bmatrix} 1 & 0 & -i\eta\cos\theta & 0 & 0 & i\beta_1\cos\theta & 0 \\ 0 & 1 & -i\eta\sin\theta & 0 & 0 & i\beta_1\sin\theta & 0 \\ i\eta\cos\theta & i\eta\sin\theta & 1 & -i\beta_1\cos\theta & -i\beta_1\sin\theta & 0 & 0 \\ 0 & 0 & -i\xi\cos\theta & 1 & 0 & i\zeta_1\cos\theta & 0 \\ 0 & 0 & -i\xi\sin\theta & 0 & 1 & i\zeta_1\sin\theta & 0 \\ i\xi\cos\theta & i\xi\sin\theta & 0 & -i\zeta_1\cos\theta & -i\zeta_1\sin\theta & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_1{}^2 \end{bmatrix}$$

$$= \delta_1{}^2(1-\eta^2)(1-\zeta_1{}^2) \neq 0.$$

The non-vanishing of the symbolic determinant gives the sufficient condition[7] for the regularization of the system of singular integral equations. The system can thus be solved for the density functions. The symbolic determinant for the other system of equations may also be shown to be non-zero in a similar manner.

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